Lecture 16

Recall: We are given a measure space $(X, \mathcal{M}, \mu)$. $L^1 = L^1(X, \mu)$ denotes the normed (metric) space of equiv. classes $[f]$ ($\sim$ means equal μ-a.e.) of integrable $f: X \to \mathbb{C}$. We still think of elements of L^1 as being functions (representatives), but only determined up to null sets. We can turn any statement "μ-a.e." to everywhere by multiplying those fcns by χ_N, where N is the null set. A fundamental result:

Dominated Convergence Thm. Let $\{f_n\}_{n=1}^{\infty} \subseteq L^1(X, \mu)$ and assume

(1) $f_n \to f$ μ-a.e. and

(2) $\exists g \in L^1(X, \mu)$ s.t. $|f_n| \leq g$ μ-a.e.

Then, $f \in L^1$ and $\int f = \lim_{n \to \infty} \int f_n$.

Pf. Per above, f is meas. (after possibly modifying it on a null set), and $|f| \le g$ $\Rightarrow \int |f| \le \int g \Rightarrow f \in L^1$.

If we write $f_n = u_n + i v_n$, $f = u + iv$, then $u_n \to u$, $v_n \to v$ a.e. and it suffices to show $\int u_n \to \int u$, $\int v_n \to \int v$.

We have, by assumption, that $g \pm u_n \in L^+$. By Fatou,

$$\int g \pm \int u \le \int g \pm \liminf_{n\to\infty} \int u_n$$

"+" $\Rightarrow$ $\int u \le \liminf_{n\to\infty} \int u_n$

"−" $\Rightarrow$ $\int u \ge \limsup_{n\to\infty} \int u_n$

$\Rightarrow$

$$\limsup_{n\to\infty} \int u_n \le \int u \le \liminf_{n\to\infty} \int u_n$$

which of course yields $\int u = \lim_{n\to\infty} \int u_n$.

Similarly for v_n, v. ☒

As a result, we have powerful convergence results not available in Riemann's theory of integrals.

Thm 1. Let $f : X \times [a,b] \to \mathbb{C}$ be s.t. $f_t := f(\cdot, t)$ integrable for each $t \in [a,b]$ and let $F : [a,b] \to \mathbb{C}$ be given by

$$F(t) = \int_X f_t(x)\, d\mu(x)$$

Then:

(i) If $\lim_{t \to t_0} f_t = f_{t_0}$ $\forall x$

$\exists g \in L^1$ s.t. $|f_t| \leq g$, $\forall x, t$, then

$$\lim_{t \to t_0} F_t = F_{t_0}$$

$$\left(\text{i.e. } \lim_{t \to t_0} \int f(x,t)\, d\mu(x) = \int \lim_{t \to t_0} f(x,t)\, d\mu(x) \right)$$

(ii) If $t \to f_t(x)$ is diff. $\forall x$ and $\exists\, g \in L^1$ s.t. $\left|\frac{\partial f_t}{\partial t}\right| \leq g \quad \forall x, t$,

then F is diff. and

$$\frac{\partial F}{\partial t} = \int_X \frac{\partial f_t}{\partial t}\, d\mu$$

$\left(\text{i.e. } \frac{d}{dt} \int f(x,t)\, d\mu(x) = \int \frac{\partial f}{\partial t}(x,t)\, d\mu(x) \right)$

<u>Rem</u>. In this thm, we are assuming that conditions hold $\forall x$ and not a.e. x. Why?. For seq., $|f_n| \leq g$ a.e. $\Rightarrow$ $\exists$ null sets N_n s.t. $|f_n| \leq g$ on N_n^c. Then $N = \bigcup_{n=1}^{\infty} N_n$ is also null and $|f_n| \leq g$ on N^c. But if $|f_t| \leq g$ a.e. for all $t \in [a,b]$, then $\exists N_t$ s.t. $|f_t| \leq g$ on N_t^c <u>but</u> $N = \bigcup_{t \in [a,b]} N_t$ need not be null. One could formulate a.e. versions but would need to be more careful.

Pf. (i) follows immediately from DCT once you recall that $\lim_{t\to t_0} h(t) = y$ $\iff \forall \{t_n\}_{n=1}^{\infty}$ s.t. $t_n \to t_0$ we have $\lim_{n\to\infty} h(t_n) = y$.

(ii) Need to check, for $t_n \to t_0$, that (any $t_0 \in [a,b]$)

$$(*) \quad \underbrace{\frac{F(t_0+t_n)-F(t_0)}{t_n} - \int \frac{\partial f}{\partial t}(x,t_0)\,d\mu(x)}_{} \to 0$$

$$\left(\int \underbrace{\left(\frac{f(x,t_0+t_n)-f(x,t_0)}{t_n} - \frac{\partial f}{\partial t}(x,t_0) \right)}_{h_n(x)} d\mu(x) \right)$$

Since $t \to f_t(x)$ is diff., by Mean Value Thm

$$\left| \frac{f(x,t_0+h)-f(x,t_0)}{h} \right| \le \sup_{\xi \in [t_0, t_0+h]} \left| \frac{\partial f}{\partial t}(x) \right|$$

$\Rightarrow$

$|h_n(x)| \le 2g$. Since $h_n \to 0$, $\forall x$, the conclusion $(*)$ follows, again by DCT. $\square$